On the Resonances of Coupled Qubit–Cavity Systems

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Abstract. We consider a system consisting of a qubit and a microwave transmission line that are coupled by a capacitor which, in turn, is modulated sinusoidally. The Unruh effect is the simultaneous production from vacuum of a pair of photons, one in the qubit and the other in the cavity. The dynamical Casimir effect is the production from vacuum of a pair of photons in the cavity. We analyze this qubit–cavity system and show that the system can be viewed as a pair of coupled quantum-mechanical oscillators and that both the Unruh effect and the dynamical Casimir effect are resonances of this coupled oscillator system. For the case where the cavity supports two propagating modes, in addition to the Unruh and dynamical Casimir effect at each of the supported modes, we predict a "paired Casimir effect," where one photon is emitted in the cavity in each of two allowed modes, at the appropriate driving frequency. We also calculate analytical approximations to the driving frequencies for all three effects.

INTRODUCTION

Quantum field theory predicts that a uniformly accelerating photodetector will detect photons in an electromagnetic vacuum which is termed the Unruh effect [1]. It also predicts that accelerating the mirror boundary of a cavity will result in the production of photon pairs in the cavity, which is termed the dynamical Casimir effect. Since the uniform acceleration required to observe the Unruh effect is very large, $\approx 10^{20}$ m/s, following other researchers [2, 3], we consider the oscillatory analog of the Unruh effect where the photodetector is moved back and forth rapidly at gigahertz frequencies, resulting in (nonuniform) acceleration of similar magnitude. In this paper, we arrive at a unified analysis of photon production from an electromagnetic vacuum based on the oscillatory Unruh and dynamical Casimir effects.

We consider the qubit–cavity system analyzed by Blencowe and Wang [3] and shown in Fig. 1. According to the predictions of the oscillatory Unruh effect [2, 3], when the coupling capacitor is driven at the appropriate frequency, it will simultaneously produce a photon in both the qubit and the cavity. The photon generated in the qubit will cause a transition from the ground to the excited state. The photon in the cavity will excite one of the cavity modes determined by the energy/frequency of the photon. Since the boundary of the cavity is also oscillating, the dynamical Casimir effect [4], where a pair of photons is produced in the cavity while the qubit remains in the ground state, should be observed, albeit at a different driving frequency.

Following the methods in [3, 5] for the single-mode case, we can show that the quantum Hamiltonian for a transmission line supporting M modes is given by $\hat{H}_\infty = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_0 = \frac{E_1 - E_0}{2}\sigma_z + \hbar\sum_{m=1}^{M}\omega_m \hat{a}_m^\dagger \hat{a}_m, \quad \text{and} \quad \hat{H}_1 \equiv \sum_{m=1}^{M} h_m = \sum_{m=1}^{M} f_m(t)\left(\sigma^+ + \sigma^-\right)\left(\hat{a}_m - \hat{a}_m^\dagger\right). \tag{1}$$

Here, E_0 and E_1 are the two lowest energy levels of the qubit, ω_m is the frequency of cavity mode m, $\hat{a}_m$ and $\hat{a}_m^\dagger$ are ladder operators, and $f_m(t) = -i\hbar g_m(1 + z(t))$, where $z(t)$ is the changing thickness between the capacitor plates, and g_m is the coupling coefficient between the qubit and cavity mode m. The detailed derivation of (1) can be found in [6]. Following [5], we have assumed that the boundary conditions on the qubit flux and cavity modes are such that the cavity modes are decoupled.

The cavity-mode terms in the Hamiltonian (1) are those of a parametric amplifier, and this analogy has been used to realize the dynamical Casimir effect using microwave cavity resonators with tunable lengths. However, we will show in our model (Fig. 1) that the combined qubit-cavity system is more analogous to a driven harmonic oscillator with the varying capacitance providing the driving term.

TIME-DEPENDENT PERTURBATION SOLUTION

We view $\hat{H}_0$ as the unperturbed Hamiltonian and $\hat{H}_1$ as a time-dependent perturbation. If we denote the cavity state as $\mathbf{n_k} = (n_{k_1}, n_{k_2}, \ldots, n_{k_M})$, the state of the unperturbed system can be represented by $|q, \mathbf{n_k}\rangle = |q, n_{k_1}, n_{k_2}, \ldots, n_{k_M}\rangle$, where $q \in \{g, e\}$ denotes the state of the qubit (ground or excited) and $n_{k_i} \geq 0$ is the total number of photons in cavity mode k_i. Thus, $N = \sum_i n_{k_i}$ is the number of photons in the cavity.

Denoting the state of the perturbed system as $\sum c_{q,\mathbf{n_k}} |q, \mathbf{n_k}\rangle$, using standard time-dependent perturbation theory (see, e.g., Eq. (5.5.17) of [7]), we have the following differential equations for the $c_{q}, \mathbf{n_k}$.

$$\dot{c}_{q,\mathbf{n_k}} = -\frac{i}{\hbar} \sum_{q'} \sum_{\mathbf{n'_k}} \sum_{m=1}^{M} \left(c_{q',\mathbf{n'_k}} \langle q, \mathbf{n_k} | h_m | q', \mathbf{n'_k} \rangle \right) e^{i\left(E_{q',\mathbf{n'_k}} - E_{q,\mathbf{n_k}}\right)t/\hbar}. \tag{2}$$

By the definition of the ladder operators $\hat{a}_m$ and $\hat{a}_m^\dagger$, and the operators σ^+ and σ^-, we can show that

$$h_m |g, \mathbf{n_k}\rangle = f_m \sqrt{n_{k_m}} |e, \mathbf{n_k} - \mathbf{1_m}\rangle - f_m \sqrt{n_{k_m} + 1} |e, \mathbf{n_k} + \mathbf{1_m}\rangle \tag{3}$$

$$\text{and} \quad h_m |e, \mathbf{n_k}\rangle = f_m \sqrt{n_{k_m}} |g, \mathbf{n_k} - \mathbf{1_m}\rangle - f_m \sqrt{n_{k_m} + 1} |g, \mathbf{n_k} + \mathbf{1_m}\rangle, \tag{4}$$

where $\mathbf{1_m}$ is the vector whose mth component is unity, and the other components are zero.

We also know that

$$E_{g,\mathbf{n_k}} = E_0 + \hbar \sum_m \omega_m \left(n_{k_m} + \frac{1}{2} \right), \quad \text{and} \quad E_{e,\mathbf{n_k}} = E_1 + \hbar \sum_m \omega_m \left(n_{k_m} + \frac{1}{2} \right). \tag{5}$$

Thus, defining $\Omega = (E_1 - E_0)/\hbar$,

$$E_{e,\mathbf{n_k} \pm \mathbf{1_m}} - E_{g,\mathbf{n_k}} = \hbar (\Omega \pm \omega_m), \text{ and } E_{g,\mathbf{n_k} \pm \mathbf{1_m}} - E_{e,\mathbf{n_k}} = \hbar (-\Omega \pm \omega_m). \tag{6}$$

Plugging these into (2), we get

$$\begin{aligned} \dot{c}_{g,\mathbf{n_k}} &= \sum_{m=1}^{M} c_{e,\mathbf{n_k}+\mathbf{1_m}} f_m \sqrt{n_{k_m} + \mathbf{1_m}} e^{it(\Omega+\omega_m)} - c_{e,\mathbf{n_k}-\mathbf{1_m}} f_m \sqrt{n_{k_m}} e^{it(\Omega-\omega_m)} \\ \dot{c}_{e,\mathbf{n_k}} &= \sum_{m=1}^{M} c_{g,\mathbf{n_k}+\mathbf{1_m}} f_m \sqrt{n_{k_m} + 1} e^{it(-\Omega+\omega_m)} - c_{g,\mathbf{n_k}-\mathbf{1_m}} f_m \sqrt{n_{k_m}} e^{it(-\Omega-\omega_m)}. \end{aligned} \tag{7}$$

Since we are dealing with weak powers, we neglect instances where three or more photons propagate in the cavity. Now, the state of the system can be written as $|q, \mathbf{0}\rangle$ when there are no photons in the cavity, $|q, \mathbf{1_n}\rangle$ when there is one photon in mode k_n, $|q, \mathbf{2_n}\rangle$ when there are two photons in mode k_n, and $|q, \mathbf{1_{m,n}}\rangle$ when there is one photon in each of the modes k_m and k_n. (Here, $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{2_n} = \mathbf{1_n} + \mathbf{1_n}$, and $\mathbf{1_{m,n}} = \mathbf{1_m} + \mathbf{1_n}$.)

We can now write our differential equations for M allowed modes of propagation:

$$\begin{aligned} \dot{c}_{g,\mathbf{0}} &= \sum_{m=1}^{M} f_m c_{e,\mathbf{1_m}} e^{it(\Omega+\omega_m)}, \qquad \dot{c}_{g,\mathbf{2_n}} = -f_n \sqrt{2} c_{e,\mathbf{1_n}} e^{it(\Omega-\omega_n)}, \quad n \in [1, M] \\ \dot{c}_{e,\mathbf{1_n}} &= f_n \sqrt{2} c_{g,\mathbf{2_n}} e^{it(-\Omega+\omega_n)} - f_n c_{g,\mathbf{0}} e^{it(-\Omega-\omega_n)} + \sum_{m \neq n} f_m c_{g,\mathbf{1_{n,m}}} e^{it(-\Omega+\omega_m)}, \quad n \in [1, M] \\ \dot{c}_{g,\mathbf{1_{n,m}}} &= -f_m c_{e,\mathbf{1_n}} e^{it(\Omega-\omega_m)} - f_n c_{e,\mathbf{1_m}} e^{it(\Omega-\omega_n)}, \quad n, m \in [1, M], m \neq n. \end{aligned} \tag{8}$$

RESONANCES FOR TWO PROPAGATION MODES

We consider the case where only two modes propagate in the cavity ($M = 2$) and write the Fock states as $|q, n_{k_1}, n_{k_2}\rangle$. We set the driving term $z(t) = 0$ in order to find the homogeneous solution to Eq. (8), which will identify the resonance frequencies of the system. In turn, we can then determine the driving frequency to excite a specific resonance.

Taking the Fourier transform of (8), we get

$$
\begin{aligned}
i\omega C_{g00}(\omega) &= g_1 C_{e10}(\omega-\Omega-\omega_1) + g_2 C_{e01}(\omega-\Omega-\omega_2) &(9)\\
i\omega C_{e10}(\omega) &= g_1\sqrt{2}C_{g20}(\omega+\Omega-\omega_1) - g_1 C_{g00}(\omega+\Omega+\omega_1) + g_2 C_{g11}(\omega+\Omega-\omega_2) &(10)\\
i\omega C_{g20}(\omega) &= -g_1\sqrt{2}C_{e10}(\omega-\Omega+\omega_1) &(11)\\
i\omega C_{g11}(\omega) &= -g_2 C_{e10}(\omega-\Omega+\omega_2) - g_1 C_{e01}(\omega-\Omega+\omega_1). &(12)
\end{aligned}
$$

(We have omitted the equations for C_{e01} and C_{g02}.) We use (9), (11), and (12) to substitute for C_{g00}, C_{g20}, and C_{g11} in (10) to get

$$
\left(\omega - \frac{g_1^2}{\omega+\Omega+\omega_1} - \frac{2g_1^2}{\omega+\Omega-\omega_1} - \frac{g_2^2}{\omega+\Omega-\omega_2}\right) C_{e10}(\omega) = g_1 g_2 C_{e01}(\omega+\omega_1-\omega_2)\left(\frac{1}{\omega+\Omega+\omega_1} + \frac{1}{\omega+\Omega-\omega_2}\right). \quad (13)
$$

Simplifying, we get an equation of the form

$$
F(\omega,\omega_1,\omega_2,\Omega,g_1,g_2)C_{e10}(\omega) = {}_1g_2 C_{e01}(\omega+\omega_1-\omega_2)\,[\omega+\Omega-\omega_1], \quad (14)
$$

where F is a polynomial of degree 4 in ω. Similarly,

$$
F(\omega,\omega_2,\omega_1,\Omega,g_2,g_1)C_{e01}(\omega) = {}_1g_2 C_{e10}(\omega+\omega_2-\omega_1)\,[\omega+\Omega-\omega_2]. \quad (15)
$$

From (15) we substitute for C_{e01} into (14) to get an equation of the form

$$
F(\omega,\omega_1,\omega_2,\Omega,g_1,g_2)F(\omega+\omega_1-\omega_2,\omega_2,\omega_1,\Omega,g_2,g_1)C_{e10}(\omega) = g_1^2 g_2^2 C_{e10}(\omega)\,[(\omega+\Omega-\omega_1)(\omega+\Omega+\omega_1-2\omega_2)]. \quad (16)
$$

The l.h.s. has terms of order g_1^2 and g_2^2. Therefore, for small g_1 and g_2 we can neglect the $g_1^2g_2^2$ term on the right and conclude that the resonance frequencies for $c_{e10}(t)$ are among the zeroes of $F(\omega,\omega_1,\omega_2,\Omega,g_1,g_2)$ and $F(\omega+\omega_1-\omega_2,\omega_2,\omega_1,\Omega,g_2,g_1)$. If we consider the g_1^2 and g_2^2 terms as perturbations, the unperturbed zeroes of $F(\omega,\omega_1,\omega_2)$ are 0, $-\Omega-\omega_1$, $-\Omega+\omega_1$, and $-\Omega+\omega_2$, and hence those of $F(\omega+\omega_1-\omega_2,\omega_2,\omega_1)$ are $\omega_2-\omega_1$, $-\Omega-\omega_1$, $-\Omega+2\omega_2-\omega_1$, and $-\Omega+\omega_2$. Therefore, the six unique possible resonance frequencies for the unperturbed $|e10\rangle$ are

$$
\omega_{R,e10} \in \{0, \omega_2-\omega_1, -\Omega+\omega_1, -\Omega-\omega_1, -\Omega+\omega_2, -\Omega+2\omega_2-\omega_1\}. \quad (17)
$$

The "zero" frequency component corresponds to steady growth of amplitude, and we should drive the system to excite this resonance. Considering (10), or its time-domain version from (8), the $c_{g,\mathbf{0}}$ dominates the r.h.s. as the other terms are small in comparison. Therefore, the driving frequency of the excitation, $z(t)$, in order to trigger resonance at $\omega_{R,e10}$ is given by

$$
\omega_{d,e10} = \omega_{R,e10} + \Omega + \omega_1 \in \{\Omega+\omega_1, \Omega+\omega_2, 2\omega_1, 0, \omega_1+\omega_2, 2\omega_2\}. \quad (18)
$$

The actual resonance frequencies will be close to these unperturbed values, and we can solve for them numerically. To find an analytical approximation to the resonance and driving frequencies for nonzero $g_1, g_2 \ll \Omega, \omega_1, \omega_2$, we use the perturbation series method: We assume the solutions to $F(\omega,\omega_1,\omega_2,\Omega,g_1,g_2) = 0$ and $F(\omega+\omega_1-\omega_2,\omega_2,\omega_1,\Omega,g_2,g_1) = 0$ are of the form $x_{00}+x_{10}g_1+x_{01}g_2+x_{20}g_1^2+x_{02}g_2^2+x_{11}g_1g_2$, expand the polynomials to second order, equate the coefficients of g_1, g_2, g_1^2, g_2^2 , and g_1g_2 to zero, and solve for the x_{ij}'s. By this method we find that the approximate driving frequency for steady resonance of the $|e01\rangle$ state is

$$
\omega_{d,\mathrm{Unruh}} = \Omega + \omega_1 + \frac{3\Omega+\omega_1}{\Omega^2-\omega_1^2}g_1^2 + \frac{1}{\Omega-\omega_2}g_2^2. \quad (19)
$$

At this driving frequency, we expect simultaneous production of pairs of photons in the qubit and in the cavity with energy $\hbar\omega_1$, and the probability of this pair production steadily increases with time. In other words, this is the driving frequency for observing the Unruh effect at ω_1.

The possible resonance frequencies for $|g20\rangle$ are obtained by shifting those of $|e10\rangle$ by $(\Omega-\omega_1)$ from (11):

$$\omega_{R,g20} \in \{\Omega-\omega_1, \Omega+\omega_2-2\omega_1, 0, -2\omega_1, \omega_2-\omega_1, 2(\omega_2-\omega_1)\}. \tag{20}$$

Now, substituting (10) in (11), we see that the possible driving frequencies for resonance of this state are obtained by adding $2\omega_1$ to these values, equivalent to adding $\Omega+\omega_1$ to the resonance frequencies of $|e01\rangle$. The perturbation series approximations are also obtained using this shift. Hence,

$$\omega_{d,\text{Casimir}} = 2\omega_1 - \frac{2}{\Omega-\omega_1} g_1^2. \tag{21}$$

The driving frequency for the dynamical Casimir effect is unaffected by the presence of the second mode, up to the accuracy of the perturbation approximation. This is unlike the Unruh driving frequency which was altered by the presence of the second cavity mode.

Finally, we consider the $|g11\rangle$ state. The resonance frequencies are obtained by shifting those of $|e10\rangle$ by $(\Omega-\omega_2)$ from (12):

$$\omega_{R,g11} \in \{\Omega-\omega_2, \Omega-\omega_1, \omega_1-\omega_2, -\omega_1-\omega_2, 0, \omega_2-\omega_1\}. \tag{22}$$

Substituting (10) in (12), we see that the driving frequencies are obtained by adding $\omega_1+\omega_2$ to these, indeed giving us the same possible unperturbed driving frequencies as for the $|e10\rangle$ and $|g20\rangle$ states.

Specifically, $\omega_1+\omega_2$ is the driving frequency for steady resonance of the $|g11\rangle$ state. At this driving frequency, a pair of photons is produced simultaneously in the cavity, one each at ω_1 and ω_2. Generalizing, if the cavity supports M modes, then we have additional resonances for the $|e\mathbf{1}_{\mathbf{m},\mathbf{n}}\rangle$ mode for $\omega_d = \omega_m + \omega_n$, $m,n \in [1,M]$. Since the case $m = n$ corresponds to the dynamical Casimir effect, we may term this the *paired Casimir effect.* There is no conceptual difficulty in allowing for a continuum of modes: we expect to see the paired Casimir effect at driving frequencies ω_d that satisfy $\omega_d = \omega_m + \omega_n$, where ω_m and ω_n are any two allowed modes.

On the same lines as above, we find that the driving frequency for the steady resonance of the $|g11\rangle$ state corresponding to the paired Casimir effect is

$$\omega_{d,\text{paired Casimir}} = \omega_1+\omega_2+\frac{1}{-\Omega+\omega_1}g_1^2+\frac{1}{-\Omega+\omega_2}g_2^2. \tag{23}$$

Numerical Results for the Two-Mode Case

For strong coupling, say, $g_1, g_2 \sim 0.1$, the driving frequencies obtained by our perturbation approximation are significantly different from the unperturbed values. We refer the reader to [6] for some numerical results to verify their accuracy. Here, we assume weak coupling coefficients, $g_1 = g_2 = 0.01$. In this case, the resonance and driving frequencies are negligibly different from their unperturbed values. Despite the weak coupling, we will see that all three effects can be clearly observed.

In Fig. 2, we plot the probabilities for each of the cavity states to illustrate the Unruh effect at ω_1. We can see that the probability of $|e10\rangle$ approaches that of the starting state $|g00\rangle$. The probabilities of the other states are negligible. Next, we plot the probabilities for each of the cavity states to illustrate the dynamical Casimir effect at ω_1 in Fig. 3(a). This is orders of magnitude weaker than the Unruh effect and can be explained as follows. The initial condition is $c_{g00}(0) = 1$, and this is the only driving term for both effects. From Eq. (10), we can see that the amplitude of the c_{e10} state which corresponds to the Unruh effect is linear in the coupling coefficient g_1, whereas by combining Eqs. (11) and (10) we see that the amplitude of the c_{g20} state is quadratic in the coupling coefficient g_1. Since we assumed $g_1 = 0.01$, we expect the relative state probability for the dynamical Casimir effect to be four orders of magnitude smaller than that of the oscillatory Unruh effect, and this is borne out by our numerical results.

Finally, we plot the probabilities for each of the cavity states to illustrate the paired Casimir effect at ω_1 and ω_2 in Fig. 3(b). This effect is also orders-of-magnitude weaker than the Unruh effect but similar in strength to the dynamical Casimir effect, which can be understood by reference to Eqs. (12) and (10).

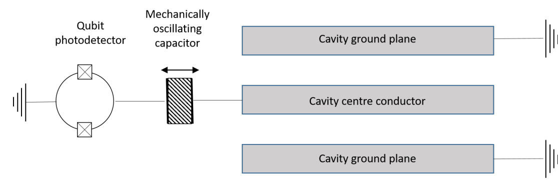


FIGURE 1. System model based on [3], consisting of a qubit photodetector and microwave cavity or transmission line coupled by a mechanically oscillating capacitor.

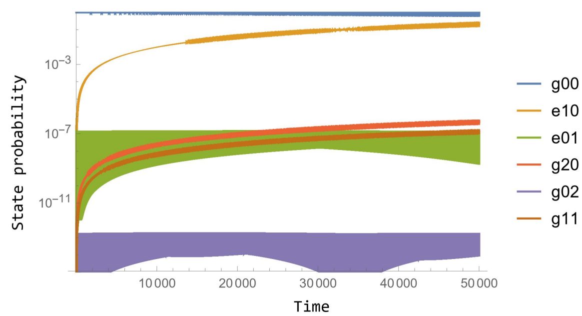
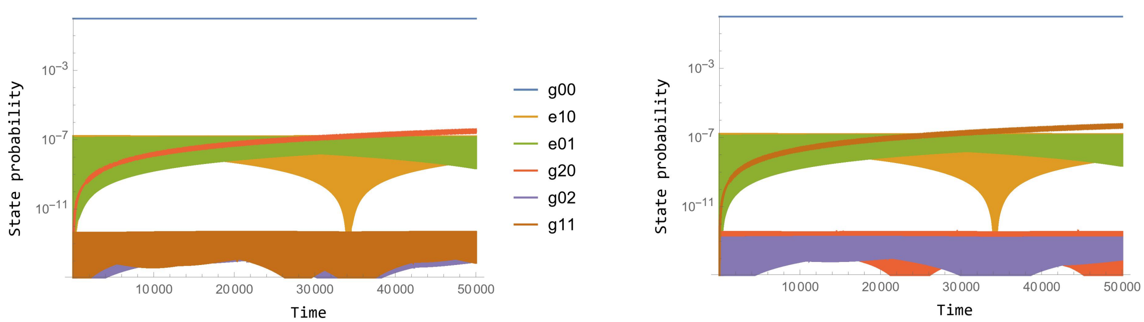


FIGURE 2. The time evolution of the state probabilities $|c_{q,n}(t)|^2$ for $\omega_1 = 9\pi$, $\omega_2 = 10\pi$, $\Omega = 6\pi$, $g_1 = g_2 = 0.01$, and $\omega_d = 15\pi$, which corresponds to the Unruh effect at ω_1, which is clearly observed.



(a) (b)

FIGURE 3.

(a) The time evolution of the state probabilities $|c_{q,n}(t)|^2$ for $\omega_1 = 9\pi$, $\omega_2 = 10\pi$, $\Omega = 6\pi$, $g_1 = g_2 = 0.01$, and $\omega_d = 18\pi$, which corresponds to the dynamical Casimir effect at ω_1: there is a non-negligible probability of occupying the state $|g20\rangle$.

(b) The time evolution of the state probabilities $|c_{q,n}(t)|^2$ for $\omega_1 = 9\pi$, $\omega_2 = 10\pi$, $\Omega = 6\pi$, $g_1 = g_2 = 0.01$, and $\omega_d = 19\pi$, which corresponds to the paired Casimir effect at ω_1 and ω_2: there is a non-negligible probability of occupying the state $|g11\rangle$.

CONCLUSIONS

We neglected states with three or more photons and analytically found the driving frequencies that would result in the oscillatory Unruh and dynamical Casimir effects for a cavity supporting two modes. In both cases we derived analytical approximations to the driving frequencies necessary to observe the oscillatory Unruh and dynamical Casimir effects. For a multimode cavity, we predicted a new paired Casimir effect where a photon is emitted in each of two different modes of the cavity.

We did not make any assumption as to the rapidity of oscillations of the coupling capacitor. Hence, our model predicts that the oscillatory Unruh and dynamical Casimir effects merely reflect the resonances of the coupled qubit–transmission line, and no relativistic effects appear to be involved. Of course, building cavities at lower than microwave frequencies is probably unfeasible.

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REFERENCES

1. W. G. Unruh, Phys. Rev. D **14**, 870–892 (1976).
2. H. Wang, M. P. Blencowe, C. M. Wilson, and A. J. Rimberg, Phys. Rev. A **99**, 053833 (2019).
3. M. P. Blencowe and H. Wang, Philos. Trans. R. Soc. London, Ser. A **378** (2020).
4. V. V. Dodonov, J. Phys. Conf. Ser. **161** (2009).
5. A. Parra-Rodriguez, E. Rico, E. Solano, and I. L. Egusquiza, Quantum Sci. Technol. **3** (2018).
6. E. N. Sivarajan, *On the Resonances of Coupled Qubit-Cavity Systems*, Senior honors thesis, Dartmouth College (2021).
7. J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed. (Cambridge University Press, 2017).