Quantum Inequalities and Particle Creation in the Presence of an External, Time-Dependent Mamaev-Trunov Potential

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Abstract. In 2011, Mr. Dan Solomon proposed a model of a quantized scalar field interacting with a time-dependent Mamaev-Trunov potential in two-dimensional Minkowski spacetime. This model is governed by the Klein-Gordon wave equation with a time-dependent potential. Mr. Solomon claims that this model violates both the classical energy conditions of special relativity and the quantum energy conditions of quantum field theory in curved spacetime. Every classical energy condition can be violated, and their natural replacements are known as quantum inequalities. Mr. Solomon attempted to prove violations of the spatial and temporal quantum inequalities, and he correctly assumed that the negative energy splits into two fluxes at the Cauchy surface, where the potential is turned off. Unfortunately, Solomon neglects the contribution to the energy density due to particle creation when the potential is turned off at time $t = 0$. In this project, we calculate the contribution to the stress energy tensor due to particle creation. We show that while the classical energy conditions are violated, the quantum energy inequalities hold, contrary to Mr. Solomon's statements.

SCIENTIFIC BACKGROUND

Mathematical Background

The mathematical foundation of quantum mechanics consists of wave functions and operators. Wave functions express the state of a system while operators represent observables. Linear algebra is the underlying mathematics of quantum mechanics, where abstract vectors represent wave functions and observables are performed as linear transformations [1]. Quantum mechanics uses Dirac notation to represent a vector as a 'ket', shown as $|a\rangle$. The dual vector for a ket is a 'bra', with the inner product 'bra-ket' written as $\langle a|b\rangle$.

An inner product space is a vector space over the real or complex numbers containing inner products or dot products. The vector spaces in which wavefunctions exist are called Hilbert spaces. Hilbert spaces are finite-dimensional and span the complex numbers [2]. A Hilbert space is a Banach space where the norm, or mapping, is an inner product. Hilbert spaces are mathematically easier to handle than general Banach spaces due to orthogonality. A Hilbert space is a complete inner product space, an example of which is the collection of square integrable functions,

$$f(x) \ \ where \int_a^b |f(x)|^2 dx < \infty, \tag{1}$$

denoted as $L_2(a, b)$. While this is a relatively small vector space, it is the Hilbert space referred to in quantum mechanics [2]. The calculations in this paper use two-dimensional Minkowski spacetime. This refers to a Euclidean manifold, with one spatial dimension and one temporal dimension, where the spacetime interval between two events does not depend on an inertial frame of reference in which the events were measured.

Another mathematical object used throughout this paper is a tensor, which is analogous to a vector-composed matrix. Tensors are arrays of functions of spatial coordinates. The most common tensor in this paper is the stress-energy tensor, with the general form of,

$$\hat{T}_{(x,t)} = \left(T^L_{tt} \ T_{tx} \ T_{xt} \ T_{xx} \right), \tag{2}$$

which describes the density and flux of matter and energy in spacetime. This is a generalization of the stress tensor of Newtonian physics. In general relativity, the Einstein tensor describes space-time curvature and the energy momentum tensor describes localized matter distribution.

Quantum Field Theory

In quantum mechanics, a single particle, with spin = ½, has two quantum states in a two-state system: one state represented as $= |\downarrow\rangle$ and the other as $= |\uparrow\rangle$. In the Copenhagen model of quantum mechanics, a particle exists in a state of superposition where the particle is simultaneously in both states. The superposition wave function for a system with two spatial states, *A* and *B*, can be written as,

$$\psi = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle|A\rangle + |\downarrow\rangle|B\rangle\right]. \tag{3}$$

A wave function describes the state of a particle. While a particle is constrained to move in one dimension, influenced by a specified force, a wave function is dependent on position for any given time. Until an act of measurement collapses the wave function, particles do not have specific dynamical properties like position or momentum [1]. Upon measurement of the system, the wavefunction spontaneously decays and the observer sees the system existing in only one of the two states. States are entangled if they are directly correlated with one another [3]. All observables in a system have corresponding wavefunctions [1]. These wavefunctions are mathematically represented by a superposition of pure time harmonic, or sinusoidal, vibrations. Multiplying the wavefunction by its complex conjugate and integrating gives the probability of finding the particle between two points for a given time. A particle's probability density is described as a wave group. The time-dependent Schrödinger equation [1], written below, can be solved to find a particle's wave function $\psi(x, t)$,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi. \tag{4}$$

In this paper, the potential V has no explicit time-dependence. The general solution to the time-dependent Schrödinger equation is a linear combination of separable solutions, a continuous sum over wavenumbers. The solution, called a wave packet, carries a range of energies and speeds. Wave packets are localized state functions consisting of a packet of waves with wavenumbers and frequencies centered around a single value k [1]. As time increases, quantum wave packets disperse, meaning that the width of the wave packet increases with time. This happens because each plane-wave component in the wave packet has a unique wave number and propagates at a different velocity [5]. Wave packets for bound states have discrete harmonic components [11] and wave packets of free electrons with initially localized position disperse over time. Wave packets of classical macroscopic objects also have dispersion times, albeit on a much longer time scale. Dispersion is an important aspect of waves and wave propagation. The dispersion relation, different for various physical systems, is the relationship between a wave's frequency and wavenumber. In quantum mechanics, the smaller the value of the spatial uncertainty σ_x, the faster electron wave packets disperse [5].

According to Heisenberg's uncertainty principle, one cannot simultaneously measure momentum and position with precise and accurate measurements for both. Instead, the more precisely one measures momentum, the less precisely one can measure position, and vice versa. With the standard deviation denoted by σ, the Heisenberg uncertainty principle for position and momentum is,

$$\sigma_x\sigma_p \geq \frac{\hbar}{2}. \tag{5}$$

Energy Conditions in Classical General Relativity

We treat spacetime as a classical curved Lorentzian manifold. This manifold is subject to the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{6}$$

If u^μ is a future-directed timelike vector and k^μ is a future-directed null vector, then the stress-tensor for matter under classical physics obeys the classical energy conditions contained in Table 1.

TABLE 1. Energy Conditions in Classical General Relativity.

Energy Condition	Inequality
Weak Energy Condition	$T_{\mu\nu}u^\mu u^\nu \geq 0$
Null Energy Condition	$T_{\mu\nu}k^\mu k^\nu \geq 0$
Strong Energy Condition	$\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)u^\mu u^\nu \geq 0$
Dominant Energy Condition	$T^\mu_\nu u^\nu \geq 0$

In classical physics, observer-measured energy density is non-negative. Thus, for all timelike vectors u^a, the matter stress-energy tensor T_{ab} obeys the weak energy condition. The weak energy condition constrains the behavior of Einstein's field equation solutions. At a critical stage during gravitational collapse, the weak energy condition makes singularity formation inevitable. Thus, gravitational mass must be positive [9].

Quantum Inequalities

A general feature of quantum field theory is the proposition every classical energy condition can be violated. This means that energy density under quantum field theory can be negative without bound. Situations yielding observer-measured negative energy density include the Casimir effect (explanation below, see Figure 1), black hole evaporation, and squeezed light states. Without placing restraints on negative energy density, it is possible to violate the cosmic censorship conjecture that every singularity must have an event horizon to hide the singularity from direct observation. In addition, it would then be possible to experience closed time-like curves or traversable wormholes, both of which are not allowed under classical physics [9]. The constraints on negative energy density come in the form of quantum inequalities, which are "natural" mathematical replacements for the classical energy conditions. Quantum inequalities constrain the duration and magnitude of negative energy fluxes. Most often, a quantum energy inequality is averaged along the worldline, or geodesic, of an inertial observer. This paper will focus on the worldline quantum inequality with mention to the spatial quantum inequality.

Cauchy Surface

A Cauchy surface is a plane in spacetime where points on the plane are spatially related but have no time difference. A spacetime possessing a Cauchy surface is inherently causal. Causality implies that the Cauchy surface can be thought of as an instant in time where the initial conditions of the plane uniquely determine future events. We study a two-dimensional spacetime where the Cauchy surface refers to the one-dimensional plane of $t = 0$. The term IN region refers to events in the causal past where $t < 0$ and the OUT region refers to the causal future where $t > 0$.

The Casimir Effect

The Casimir effect is a physical force due to the presence of a quantized field. Mechanically speaking, it is the attractive force between two parallel perfectly conducting plates held at a close separation distance. The force arises from the quantum and thermal vacuum fluctuations of the electromagnetic field [15].

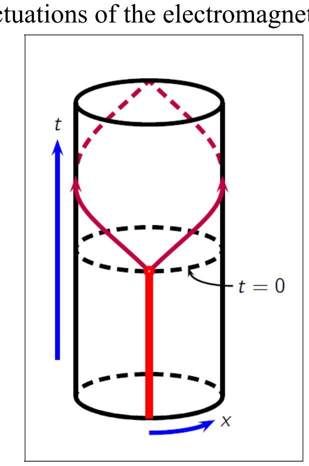

FIGURE 1. The C1asimir effect (red) in two-dimensional spacetime.

The Casimir effect for a two-dimensional, potential-free cylinder spacetime is displayed in Figure 1. The expression for a cylinder spacetime with no potential for the OUT region ($t > 0$) is,

$$\langle \tilde{0}_L \left| T_{00} \right| \tilde{0}_L \rangle_{Ren.} = \left(\frac{-\pi}{6L^2} + \frac{1}{4lL} \right) \delta_{\mu\nu}. \tag{7}$$

The expression for a cylinder spacetime for the IN region ($t < 0$) is,

$$\langle 0_L \left| T_{00} \right| 0_L \rangle_{Ren.} = \left(\frac{-\pi}{6L^2} + \frac{A}{L^2} \right) \delta_{\mu\nu}. \tag{8}$$

which is true for all locations except that of the delta-function potential.

A motivating problem for this research is calculating the Casimir effect for a scalar field in the presence of delta-type potentials. The "Casimir problem" refers to the response of a quantum field's fluctuations in response to externally imposed boundary conditions. Physically, there are no interactions strong enough to enforce a boundary condition on every frequency of a fluctuating field. Graham et al. explore a physical model of this situation [12].

Wightman Axioms

Despite the long history of quantum field theories, there are no rigorous descriptions of the structure of quantum field theories. Quantum field theory is often described as the quantization of classical field theories. Quantum field theories that use axioms, known as axiomatic quantum field theories, take a more systematic approach. Axiomatic quantum field theory can explain the transition from Minkowski spacetime to Euclidean spacetime, therefore explaining the transition from relativistic quantum field theory to Euclidean.

Wightman fields are operator valued distributions satisfying the Wightman Axioms. The Wightman functions, which are used in this paper, are the functions that correlate to Wightman fields. Wightman quantum field theory consists of the space of states (the projective space of a complex Hilbert space), the vacuum vector, a unitary representation of the Poincaré group (which is the group of Minkowski spacetime isometries, or length-preserving linear transformations), and field operators. This data collectively satisfies the three Wightman Axioms of covariance, locality, and spectrum condition [15].

Green's Functions

Green's theorem is often used in electrostatics problems involving finite regions of space with bounding surfaces and prescribed boundary conditions. This is because it provides mathematical tools to handle boundary conditions [6]. In previous papers, Green's functions are used to derive the energy conditions with a closed boundary and Cauchy boundary conditions [7]. One begins with the wave equation, which typically has the basic structure,

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(x, t), \tag{9}$$

where $f(x,t)$ is a known source distribution and c is the velocity of wave propagation in the medium. The solution to the wave equation can be represented as a sum of mode functions. The Euclidean two-point function is equal to the sum of the mode functions and is the analogue of a Feynman Green's function, G(**x**, **x'**), for the Lorentzian metric. Thus, to solve the wave equation, it is helpful to first find a Green's function.

The general solution for a Green's function is comprised of the advanced and retarded Green's functions, $G^{(-)}(x; x')$, $G^{(+)}(x; x')$, respectively. The general solution for the Green's function is,

$$G(R) = AG^{(+)}(R) + BG^{(-)}(R), \tag{10}$$

where $R = |R|$, where $R = x - x'$ given the vectors of points x and x'. Coefficients A and B depend on the boundary conditions of the given problems. $G^{(+)}$, the in-traveling wave, exhibits the causal behavior associated with a wave disturbance. The term $AG^{(+)}(R)$ represents a diverging spherical wave that propagates from the origin. Likewise, $G^{(-)}$ is the advanced Green's function where the term $BG^{(-)}(R)$ represents a converging spherical wave traveling toward the origin [6].

The time-dependent Green's functions for a nondispersive medium are,

$$G^{(\pm)}(R, \tau) = \frac{1}{R} \delta \left(\tau \mp \frac{R}{c} \right), \tag{11}$$

where the Green's functions are dependent on the relative distance, $R = x - x'$, and the relative time, $\tau = t - t'$, between source and observation point. The delta function's argument shows that an effect, observed at point **x** at time *t,* is caused by the action of a source located a distance **R** away occurring at an earlier (retarded) time. The time difference, represented as $\frac{R}{c}$, is the time of propagation of the disturbance between the two points. To solve the wave function, one can integrate the Green's function $G^{(\pm)}\left(x, t; x', t'\right)$ and source distribution $f(x', t')$,

$$\Psi^{(\pm)}(x, t) = \iint G^{(\pm)}\left(x, t; x', t'\right) f(x', t') d^3x' dt'. \tag{12}$$

This equation applies to a source distribution localized in time and space. To apply the above equation to a definite physical problem, one may add solutions to the homogeneous equation [6].

By calculating the solutions to the wave equation, one can construct the quantum inequalities by summing the solution mode functions. Thus, by knowing the two-point function for the given spacetime, one can calculate the quantum inequalities by Euclideanizing and taking the necessary derivatives of the two-point function [7].

PROJECT INTRODUCTION

In classical physics, there exist classical energy conditions that mathematically constrain energy density in space and time to be nonnegative (Table 1). These come from the observation that mass is only positive in value. However, in quantum physics, energy density can be negative. Quantum inequalities are natural replacements for classical energy conditions and are local constraints on the extent and magnitude of negative energy density in spacetime. In quantum field theory, the energy density can be calculated from the Wightman two-point function,

$$\rho(x, t, x', t') = \tfrac{1}{2}\left(\partial_t \partial'_t + \partial_x \partial'_x\right) G\left(x, t, x', t'\right) . \tag{13}$$

Mamaev and Trunov proposed a novel method to calculate the Casimir effect, explained below. They calculated the vacuum expectation value of the stress-energy tensor for a relativistic quantum field theory, where the quantum field interacts with an externally applied potential featuring two Dirac delta functions. A paper by Solomon asserts that this violates the spatial energy condition. However, Solomon did not include the energy contribution of particle creation to the stress-energy tensor. In contrast, in this paper we include the effects of particle creation.

We propose calculating the energy contribution to the stress-energy tensor caused by particle creation at the Cauchy surface, where $t = 0$. To mathematically prove the correctness of this approach, we examine two-dimensional Minkowski spacetime in the presence of an external, time-dependent Mamaev-Trunov potential. For simplicity, we are using the relativistic Klein-Gordon-Fock wave equation for massless, spinless particles,

$$\frac{\partial^2 \varphi}{c^2 \partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + V(x, t)\varphi = 0. \tag{14}$$

The Klein-Gordon-Fock relativistic wave equation originates from the Schrödinger equation and describes energy and momentum. The equation is second order in space and time and describes the dynamics of a boson particle. We use this equation because it is mathematically simpler to use than the Dirac equation for relativistic particles with half integer spin.

The scalar quantum field with the time-dependent potential is,

$$\frac{\partial^2 \varphi}{c^2 \partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + 2\lambda V(x)\,\theta(-t)\varphi = 0, \tag{15}$$

with the Mamaev-Trunov-type potential given by,

$$V(x) = \left[\delta\left(x - \tfrac{a}{2}\right) + \left(x - \tfrac{a}{2}\right)\right]. \tag{16}$$

The potential is two delta-functions barriers separated by a distance a and centered at the origin. The presence of the delta-functions causes a constant, negative-energy Casimir effect in the empty region between them and zero energy density outside of the delta-functions [3]. When the potential is shut off, the negative-energy Casimir effect becomes dynamical, and begins to move left- and right-ward in the spacetime.

Substituting (10) into (9) yields,

$$\frac{\partial^2 \varphi}{c^2 \partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + \lambda\left[\delta\left(x + \tfrac{a}{2}\right) + \delta\left(x - \tfrac{a}{2}\right)\right]\Theta(-t)\,\varphi = 0, \tag{17}$$

where λ is the coupling constant, δ is the Dirac Delta function, and Θ is the unit step function.

Results of Mamaev And Trunov

Mamaev and Trunov, in their 1981 paper [11], proposed a novel method to calculate the Casimir effect by calculating the vacuum expectation value of the stress-energy tensor for a relativistic quantum field theory. Mamaev and Trunov calculate the vacuum energy-momentum tensor for bounded manifolds without necessitating a cut-off value. They replace impenetrable boundaries with localized potentials dependent on λ. This method uses a quantum field that interacts with an externally applied potential- the “Mamaev-Trunov potential” used in this paper. Mamaev and Trunov calculate the well-behaved difference between the expectation value of the stress-energy tensor with and without a potential. By taking the limit of the coupling strength as the potential approaches infinity, they calculate the traditional Casimir effect.

The quantum field, $\Phi(x,t)$, in this model utilizes the Klein-Gordon-Fock equation with a time-independent potential,

$$\left[\partial_t^2 - \partial_x^2 + \lambda V(x)\right]\Phi(x,t) = 0. \tag{18}$$

Mamaev and Trunov determine the kinetic energy density, the part of the energy density that does not explicitly dependent on the scalar potential, of a massless scalar field in one-dimensional spacetime. They use a simple potential comprised of two Dirac delta functions that have a separation distance of a and are centered around the origin. The potential is given by,

$$V(x) = \delta\left(x + \tfrac{a}{2}\right) + \delta\left(x - \tfrac{a}{2}\right). \tag{19}$$

Teis yields a negative result for kinetic energy density in the region between $\frac{-a}{2}, \frac{a}{2}$, shown in grey in Figure 2. Next, they calculate the renormalized vacuum expectation value of the energy-density operator. This is expressed

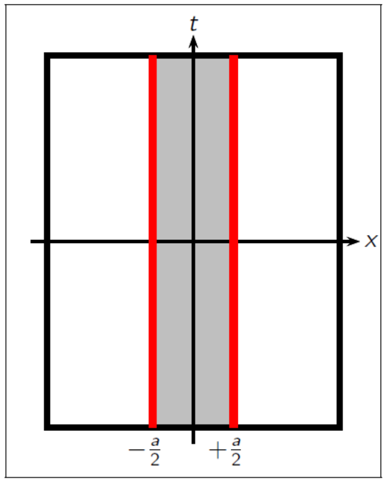

FIGURE 2. Mamaev and Trunov's solution for the Casimir effect (red).

mathematically as,

$$\langle\Omega\left|T_{00}\right|\Omega\rangle_{ren}(x) = \varepsilon\left[\Theta\left(x + \tfrac{a}{2}\right) - \Theta\left(x - \tfrac{a}{2}\right)\right], \tag{20}$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$, and ε_1 is the contribution to the vacuum expectation value due to the "odd" mode solutions and ε_2 is the contribution to the vacuum expectation value due to the "even" mode solutions.

An approximation of the value of the Casimir energy density is given by,

$$\varepsilon(\lambda, a) = -\frac{\lambda}{2\pi a}\int_0^\infty \frac{dy\, ye^y}{ye^y + \sinh\frac{\lambda a}{2}\sinh y} + \frac{\lambda}{2\pi a}\int_0^\infty \frac{dy\, ye^y}{ye^y + \cosh\frac{\lambda a}{2}\cosh y} \simeq -\frac{3\pi\lambda}{4a(18\lambda a + 7\pi^2)}. \tag{21}$$

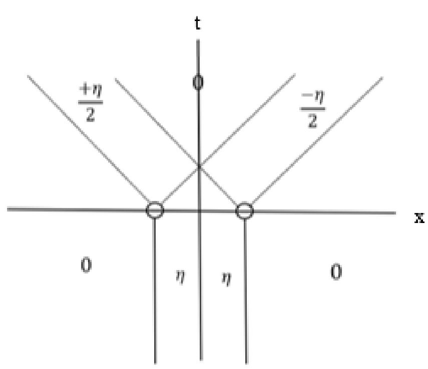

FIGURE 3. Our competing model. The difference between Mr. Solomon's model and ours is that our model includes boundary conditions for the energy fluxes.

Results of Graham et al.

Graham et al. [12] explore a physical model of the Casimir effect by coupling a fluctuating field to a smooth background potential where the boundary condition is implemented in a certain limit. Graham et al. develop new methods to compute renormalized energy densities and single loop quantum energies. The former process is the method we use in this paper to compute the renormalized energy density. The approach of Graham et al. uses scattering data to compute Green's functions for time-independent background fields. Their calculation is useful for the numerical study of single limits since it both avoids oscillating terms as well as exponentially growing and decaying terms.

We examined the results of Graham et al. [12] to learn from their general approach to calculating a complete renormalization of the expectation value of energy density for a scalar field in the presence of a potential. Graham et al. calculate a massive scalar field in Minkowski spacetime that obeys the wave equation,

$$\left[\partial_t^2 - \partial_x^2 + m^2 + \lambda\left(\delta\left(x - \tfrac{a}{2}\right) + \delta\left(x + \tfrac{a}{2}\right)\right)\right]\Phi(x,t) = 0. \tag{22}$$

Graham et al. renormalize by identifying divergent contributions to the Casimir energy. They separate the expectation value of the energy density operator into three parts,

$$\langle\Omega\left|T_{00}\right|\Omega\rangle_{Ren.} = \tfrac{1}{2}\left(\bar{\epsilon} + \epsilon_{FD} + \epsilon_{CT}\right), \tag{23}$$

where,

$$\bar{\epsilon}(x,a,\lambda) = \int_m^\infty \frac{dt}{\pi}\frac{\eta(it,x)}{\sqrt{t^2-m^2}}, \tag{24}$$

and,

$$\epsilon_{FD}(x,a,\lambda) + \epsilon_{CT}(x,a,\lambda) = \frac{\lambda}{2\pi}\delta\left(|x| - \tfrac{a}{2}\right) - \frac{m^2\lambda}{2\pi}\int_m^\infty \frac{dt}{t\sqrt{t^2-m^2}}\left(e^{-2\left|x-\frac{a}{2}\right|t} + e^{-2\left|x+\frac{a}{2}\right|t}\right). \tag{25}$$

The vacuum expectation value of the energy density is then calculated as,

$$\langle\Omega\left|T_{00}\right|\Omega\rangle_{Ren.} = \left(-\infty + \frac{\lambda}{4\pi}\right) + \left[\delta\left(x - \tfrac{a}{2}\right) + \delta\left(x + \tfrac{a}{2}\right)\right] + \{0\, for\ |x| > a/2,\ \varepsilon(\lambda,a)\, for\ 0 \le |x| < a/2. \tag{26}$$

Results of Flanagan

We examine the results of Flanagan [9] to calculate energy density of the stress-energy tensor by using bounds. Flanagan analyses the behavior of the renormalized expected stress-energy tensor in two-dimensional Minkowski spacetime for a free massless scalar field. Flanagan calculates the optimal lower bound and characterizes its respective state. Next, Flanagan calculates the lower bound for an arbitrary smooth positive weighting function. Flanagan's definition for the temporally sampled energy density for fixed spatial position x_0 is,

$$\varepsilon_T[f] = \int_{-\infty}^{\infty} \langle\omega\left|: T_{00} :\right|\omega\rangle(x_0,t) f(t)\, dt, \tag{27}$$

where $T_{\mu\nu}$ is the stress-energy operator and $f(v)$ is a non-negative normalized sampling function. Flanagan then defines the spatially sampled energy density as,

$$\int_{-\infty}^{\infty} \langle\rho\rangle_{Ren.}(\tau) f(\tau)\, d\tau \ge -\frac{1}{24\pi}\int_{-\infty}^{\infty} \frac{f(v)^2}{f(v)}\, dv. \tag{28}$$

Results of Solomon

In 2011, Dan Solomon published a paper [10] claiming that his model would violate Flanagan's two-dimensional worldline quantum inequality [9],

$$\int_{-\infty}^{\infty} \rho(\tau) f(\tau)\, d\tau \ge -\frac{1}{24\pi}\int_0^{\infty} \frac{|f'(v)|^2}{f(v)}\, dv, \tag{29}$$

where $\rho(\tau)$ is the energy density, $f(\tau)$ is a sampling function, and $f(v)$ is a non-negative normalized sampling function.

Solomon derived that the negative energy between the two delta-functions splits into two fluxes of negative energy, one moving left and one moving right as shown in Fig. 2. For an observer sitting to the right of the region of the potential, the negative energy-density integrated against Solomon's test function results in the inequality,

$$\frac{-\eta}{2}\left(\frac{3a^5}{8t_0^5}+\frac{5a^3}{2t_0^3}\right)\geq -\frac{5}{3\pi t_0^2}, \tag{30}$$

which indeed violates the worldline quantum inequality in two dimensions. Solomon uses a similar technique to violate the spatial quantum inequality. If the fluxes shown in Figure 2 are correct, then the quantum inequalities would fail, demonstrating a mathematical flaw in a fundamental property of quantum physics.

However, turning off the Mamaev-Trunov potential at the Cauchy surface, where $t=0$, causes a change in the background spacetime. This leads to particle creation that causes excitations of the "standard" modes in the "OUT" region of spacetime, where $t>0$. Solomon neglects the fact that shutting off the potential at time $t=0$ results in particle creation. Not incorporating particle creation results in an incorrect value for the energy contribution to the stress-energy tensor, eventually leading to a different result.

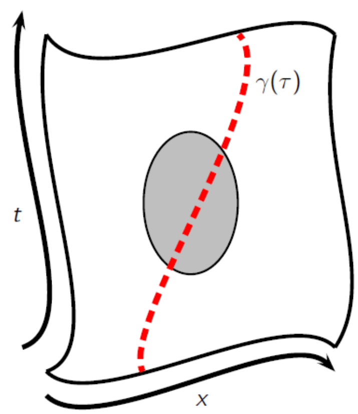

FIGURE 4. A two-dimensional universe with a timelike geodesic (red line) passing through a region of space encompassing negative energy density (grey region).

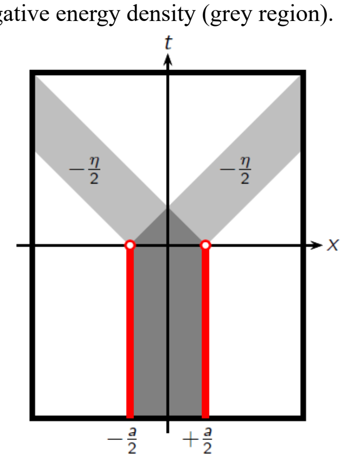

FIGURE 5. Solomon's Model.

Previous Work on Topic

A previous paper published by Pfenning [13] studies a massless, quantized scalar field in the presence of an external, time-dependent Mamaev-Trunov potential for a single delta-function in two-dimensional cylinder spacetime, illustrated in Figure 6. As in this project, the quantum field is governed by the Klein-Gordon-Fock wave equation.

The renormalized expectation value of the stress-energy tensor for the IN region is,

$$\langle 0_L \left| T^{IN}_{\mu\nu} \right| 0_L \rangle_{Ren} = \left(-\frac{\pi}{6L^2} + \frac{B-C}{L^2} \right) \delta_{\mu\nu}, \tag{31}$$

and for the OUT region is,

$$\langle 0_L \left| T^{OUT}_{\mu\nu} \right| 0_L \rangle_{Ren} = \frac{\lambda}{4\pi} \left[\delta\left(t+x\right) + \delta(t-x) \right] \delta_{\mu\nu} + \frac{\lambda}{4\pi} \left[\delta\left(t+x\right) - \delta(t-x) \right] [0\ 1\ 1\ 0\ ]\,. \tag{32}$$

Shutting the potential off at time $t = 0$ creates two positive energy pulses, shown in Figure 7. The pulses have magnitude $\frac{\lambda}{4\pi}$; one pulse moves in the $-x$ direction and the other in the $+x$ direction.

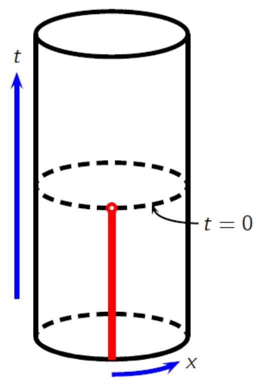

FIGURE 6. Pfenning's model of cylinder spacetime in the presence of a potential.

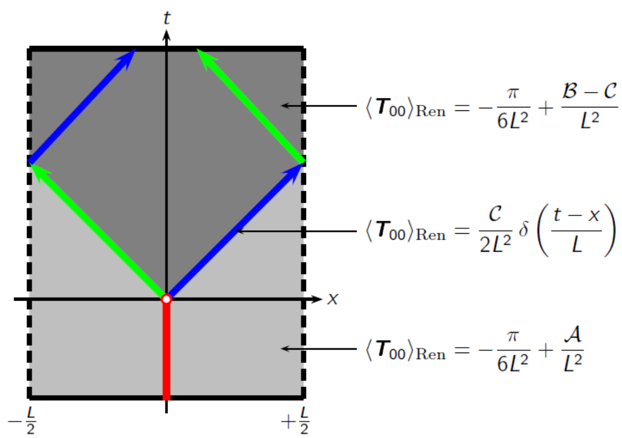

FIGURE 7. Findings from Dr. Pfenning's model of cylindrical spacetime.

CALCULATIONS

This paper mathematically demonstrates that there exist excitations of the "standard" modes of the OUT region of spacetime when the potential is turned off. The first task is to solve two initial value problems for Cauchy data (initial data) at time $t = 0$. Doing so involves calculating the Fourier decomposition of plane waves from the causal past (time $t < 0$) to the Cauchy data surface. This step was calculated by Mamaev and Trunov in their 1982 paper [14]. The antisymmetric mode solution to the Klein-Gordon equation is,

$$\Phi^{-}_{k1}\left(x,t\right) = \frac{1}{\sqrt{2\pi w}} e^{-iwt} \sin sin\ \left(kx\right)\ , \tag{33}$$

and the symmetric mode solution is,

$$\Phi^{-}_{k2}\left(x,t\right) = \frac{1}{\sqrt{2\pi w}} e^{-iwt} cos\left(k\left|x\right| + \delta\right). \tag{34}$$

We calculate the Cauchy data as,

$$\Phi^{-}_{k1}\left(x,0\right) = \frac{1}{\sqrt{2\pi w}} \sin sin\ \left(kx\right) \tag{35}$$

$$\partial_t \Phi^{-}_{k1}\left(x,0\right) = \frac{-iw}{\sqrt{2\pi w}} \sin sin\ \left(kx\right)\ , \tag{36}$$

for the antisymmetric modes and,

$$\Phi^{-}_{k2}(x,t) = \frac{1}{\sqrt{2\pi w}} e^{-iwt} cos\,(k\,|x| + \delta) \tag{37}$$

$$\partial_t \Phi^{-}_{k1}(x,0) = \frac{-iw}{\sqrt{2\pi w}} cos\,(k\,|x| + \delta)\,, \tag{38}$$

for the symmetric modes.

The next task is to determine the Fourier coefficients, $\overline{\alpha(\kappa)}$ and $\beta(\kappa)$. We begin by taking the Fourier transform of the functions at the Cauchy surface. Our calculated value for the Fourier coefficients is,

$$\overline{\alpha(\kappa)} = \frac{1}{i2\sqrt{2}}\left(\sqrt{\frac{\omega}{w}} + \sqrt{\frac{w}{\omega}}\right)(\delta(\kappa - k) - \delta(\kappa + k)) \tag{39}$$

$$\beta(\kappa) = \frac{1}{i2\sqrt{2}}\left(\sqrt{\frac{\omega}{w}} - \sqrt{\frac{w}{\omega}}\right)(\delta(\kappa - k) - \delta(\kappa + k))\,, \tag{40}$$

for the antisymmetric modes and,

$$\overline{\alpha(\kappa)} = \frac{1}{2\sqrt{2\pi}}\left(\sqrt{\frac{w}{\omega}} + \sqrt{\frac{\omega}{w}}\right)\{\pi cos\delta[\delta(\kappa - k) + \delta(\kappa + k))] - \frac{2ksin\delta}{k^2-\kappa^2}\} \tag{41}$$

$$\beta(\kappa) = \frac{1}{2\sqrt{2\pi}}\left(\sqrt{\frac{w}{\omega}} - \sqrt{\frac{\omega}{w}}\right)\{\pi cos\delta[\delta(\kappa - k) + \delta(\kappa + k))] - \frac{2ksin\delta}{k^2-\kappa^2}\}\,, \tag{42}$$

for the symmetric modes. Notice that the Fourier coefficients for the symmetric mode are more complicated than the Fourier coefficients for the antisymmetric mode.

We next calculate the continued evolution of each mode for t > 0, using the plain wave mode,

$$\Psi_\kappa(x,t) = \frac{1}{\sqrt{4\pi\omega}} e^{i(\kappa x - \omega t)}, \tag{43}$$

and the generic solution,

$$\Psi\;(x,t) = \int_{-\infty}^{+\infty} [\overline{\alpha(\kappa)}\Psi_\kappa(x,t) - \beta(\kappa)\overline{\Psi_\kappa(x,t)}]d\kappa \tag{44}$$

where $\overline{\alpha(\kappa)}$ and $\beta(\kappa)$ are the Fourier coefficients. The continued evolution of each mode is,

$$\Psi^{even}_{(x,t)} \;=\; \frac{1}{\sqrt{2\pi w}}\{cos\delta \cos cos\;(kx)\;e^{-iwt} \tag{45}$$

$$- \frac{k}{\pi} sin\delta \int_{-\infty}^{\infty} \frac{d\kappa}{k^2-\kappa^2}[\cos cos\;(\kappa x - \omega t)\;+ i\frac{w}{\omega}\sin sin\;(\kappa x - \omega t)\;]\} \tag{46}$$

$$\Psi^{odd}_{(x,t)} \;=\; \frac{1}{\sqrt{2\pi w}} \sin sin\;(kx)\;e^{-iwt}. \tag{47}$$

Next, we combine the Fourier evolution of the modes and renormalize by subtracting the same expression for $\lambda = 0$. We then take the derivative of the Wightman function to find the unrenormalized density,

$$\rho(x,t,x',t) = \frac{1}{2}\left(\partial_t\partial'_t + \partial_x\partial'_x\right)G\left(x,t,x',t'\right), \tag{48}$$

where $G\left(x,t,x',t'\right)$ is the Wightman function, explained in the 'Wightman Axioms' section.

We then calculate the Fourier transform for IN region plane waves at the Cauchy surface to find the Fourier coefficients of plane waves to the causal future. The result is,

$$\varphi^{OUT}(x,t) = \int_{-\infty}^{\infty} dk[a(k)\,\psi(k,x,t) + b(k)\,\psi(k,x,t)], \tag{49}$$

where *a(k)* and *b(k)* are the Fourier coefficients for the IN region. We find that the number of created particles is proportional to $|b(k)|^2$.

We next calculate the mode contributions for the entirety of our two-dimensional spacetime, and find that the antisymmetric mode contribution is,

$$G^{+}_{Anti}\left(\bar{x};\bar{x}'\right) = \frac{1}{2\pi}\int_0^\infty dk\frac{1}{\sqrt{w}} si\,(kx)\;e^{-iwt}\frac{1}{\sqrt{w}} si\,(kx')\;e^{+iwt'}, \tag{50}$$

and the symmetric mode contribution is,

$$G^{+\,REN}_{Sym}(\bar{x},\bar{x}') = \frac{1}{2\pi}\int_0^\infty \frac{\partial k}{w}[cos\,(\kappa\,|x| + \delta)\,cos\left(\kappa\left|x'\right| + \delta\right) - cos\,(\kappa\,|x|)\,cos(\kappa\left|x'\right|]e^{-iw(t-t')}. \tag{51}$$

Note that these results display the renormalized positive frequency Wightman function. The next step is to calculate the expectation value of energy density in the OUT region. We calculate the vacuum expectation value of the stress-energy tensor to be,

$$< \left| T_{tt} \right| 0 >_{ren} (x,t) = \frac{\lambda}{8\pi} [\delta \left(x + t - \frac{a}{2} \right) + \delta \left(x + t + \frac{a}{2} \right) - \frac{\eta}{2} \left[\Theta \left(x + t + \frac{a}{2} \right) - \Theta \left(x + t - \frac{a}{2} \right) \right] + \\ [\frac{\lambda}{8\pi} \delta \left(x - t - \frac{a}{2} \right) + \delta \left(x - t + \frac{a}{2} \right) - \frac{\eta}{2} \left[\Theta \left(x - t + \frac{a}{2} \right) - \Theta \left(x - t - \frac{a}{2} \right) \right] . \tag{52}$$

Choosing a stationary geodesic at x_0 which is either to the right or the left of the potential, we determine that the energy density is,

$$\rho_{ren}(t) = \langle 0 \left| T_{tt} \right| 0 \rangle_{ren} (x_0, t) . \tag{53}$$

The left-hand side of the quantum inequality comes from the integration of this expression against Solomon's test function, i.e.

$$LHS = \int_0^\infty \rho_{ren}(t) f(t)\, dt, \tag{54}$$

where,

$$f(t) = \frac{30}{t_0^5} t^2 (t - t_0)^2 . \tag{55}$$

Our resulting inequality is,

$$- \frac{5}{3\pi t_0^2} \frac{\lambda}{4\pi a} \left(\frac{27a^5}{10t_0^5} + \frac{25a^3}{4t_0^3} \right) \geq - \frac{5}{3\pi t_0^2} . \tag{56}$$

This result, unlike Solomon's, is always true, as the left-hand side is positive for all values of a and t_0. The positive energy pulse on the leading and trailing edges of the traveling negative-energy regions overwhelms the negative energy contributions in the quantum inequality. Therefore, particle creation and the positive energy associated with it play a dominant role in the dynamical model and the spatial quantum inequality is not violated.

CONCLUSION

Classical energy conditions are mathematical constraints on energy conditions in space and time. In classical physics, the energy conditions constrain energy density to be nonnegative because mass is positive in value. In quantum physics, however, energy density can be negative. The replacements for classical energy conditions, quantum inequalities locally constrain the magnitude and extent of negative energy density in spacetime. Mamaev and Trunov generated a method to calculate the Casimir effect, a physical force arising due to the presence of a quantized field, by calculating the vacuum expectation value of the stress-energy tensor for a relativistic quantum field theory and using a double delta function potential. A paper by Dan Solomon claims that this violates the spatial energy condition; however, Solomon neglected the energy contribution of particle creation to the stress-energy tensor. In this paper, we include the effects of particles created after shutting off the potential at the Cauchy surface and re-calculate the energy contributions to the stress-energy tensor. We do so in a two-dimensional Minkowski spacetime with an external, time-dependent Mamaev-Trunov potential. We find that the spatial quantum inequality is not violated.

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